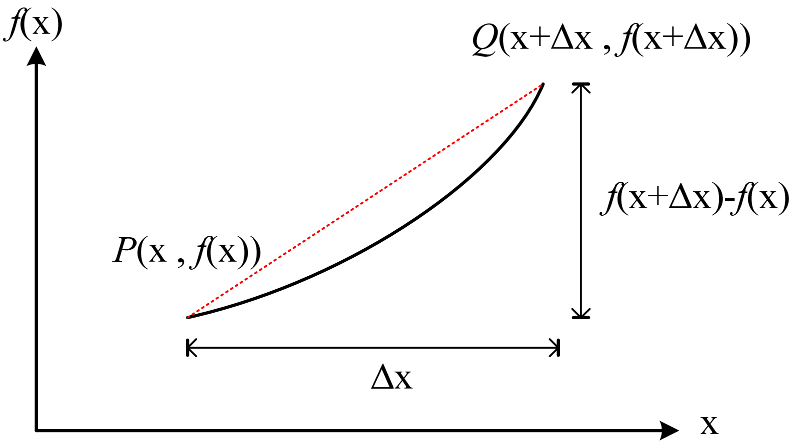
Chapter 1 Limist

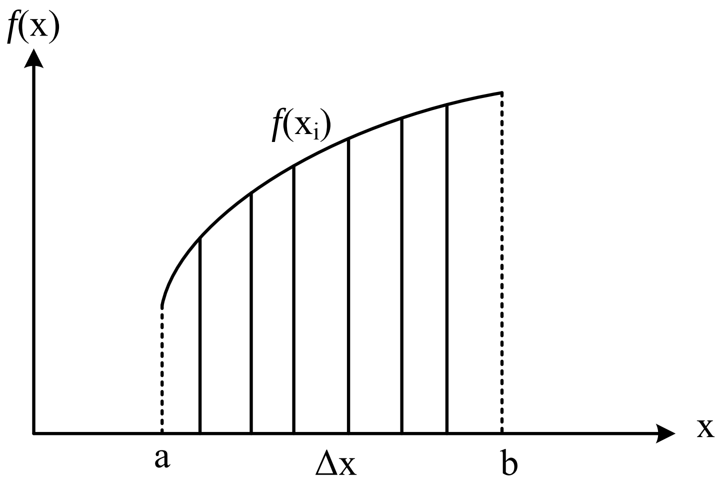
- Tangent line problem



$$m_{sec} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\therefore m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

- Area problem



$$\Delta x = \frac{b-a}{n}$$

$$\Delta A \fallingdotseq f(x_i) \times \Delta x = f(x_i) \times \frac{b-a}{n}$$

$$A = \lim_{n \to 0} f(x_i) \times \frac{b-a}{n} = \int_a^b f(x) \times dx$$

- Common type of behavior associated with nonexistence of a limit

  (1) f(x) approaches a different number

$$\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$$

  (2) f(x) increases or decreases without bound as $x \to c$

  (3) f(x) oscillates between two fixed values as $x \to c$

- Some type of limits

(1) limits of composite function

$$\lim_{x\to c} g(x) = L \ , \lim_{x\to L} f(x) = f(L) \ \text{then} \ \lim_{x\to c} f(g(x)) = f(L)$$

ex1. $g(x) = \sqrt{\dfrac{x}{x^2+1}}$, $f(x) = \sqrt{x^2+2}$, find $\lim_{x\to 1} f(g(x))$

Sol：

$$\lim_{x\to 1} g(x) = \sqrt{\frac{1}{2}} \quad \therefore \lim_{x\to 1} f(g(x) = f(\sqrt{1/2}) = \sqrt{(1/2)+1} = \sqrt{\frac{5}{2}}$$

(2) Cancellation and Rationalization Techniques

ex2. $\lim_{x\to -3} \dfrac{x^2+x-6}{x+3}$

Sol：

$$\text{original Equation} = \lim_{x\to -3} \frac{0}{0} \quad (\text{Indeterminate form})$$

$$\therefore \lim_{x\to -3} \frac{(x+3)(x-2)}{x+3} = -5 \quad (\text{by partial fractions})$$

(3) Rationalization Technique

ex3. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

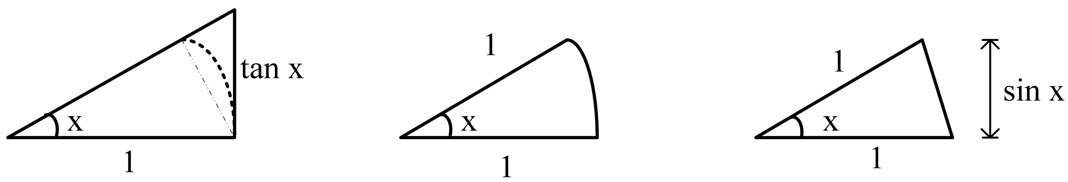
$$\lim_{x\to 0}\frac{1-1}{0}\ \text{(Indeterminate form)}$$

Sol： $\therefore O.E.=\lim_{x\to 0}\frac{\left(\sqrt{x+1}-1\right)\times\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0}\frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$

$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+1}+1\right)}=\frac{1}{2}$$

(4) The squeeze Theorem

(1)



$$A_1=\frac{1}{2}\tan x \qquad A_2=\frac{x}{2} \qquad A_3=\frac{1}{2}\sin x$$

$$(\pi r^2\times\frac{x}{2\pi}=\frac{r^2\pi}{2}=\frac{x}{2})$$

$\therefore\ \tan x<x<\sin x$

$\frac{1}{\tan x}>\frac{1}{x}>\frac{1}{\sin x}$ $\qquad \lim_{x\to 0}\cos x=1$

$\frac{\sin x}{\tan x}>\frac{\sin x}{x}\geq\frac{\sin x}{\sin x}$ $\qquad \lim_{x\to 0}1=1$

$\cos x\geq\frac{\sin x}{x}\geq 1$ $\qquad \lim_{x\to 0}\frac{\sin x}{x}=1$

(2)

$$\lim_{x\to 0}\frac{1-\cos x}{x} \quad (\text{indeterminate form})$$

$$\text{O.E.}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{\sin x}{1+\cos x}=1\times 0=0$$

ex.5. $\lim_{x\to 0}\frac{\tan 4x}{\tan 3x}$

Sol： $\frac{0}{0}$ (indeterminate form)

$$\therefore \text{O.E.}=\lim_{x\to 0}\frac{\frac{\sin 4x}{\cos 4x}}{\frac{\sin 3x}{\cos 3x}}=\lim_{x\to 0}\frac{\sin 4x}{\sin 3x}\times\frac{\text{cox } 3x}{\sin 4x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin 4x}{4x}\times 4x}{\frac{\sin 3x}{3x}\times 3x}\times\frac{\cos 3x}{\cos 4x}$$

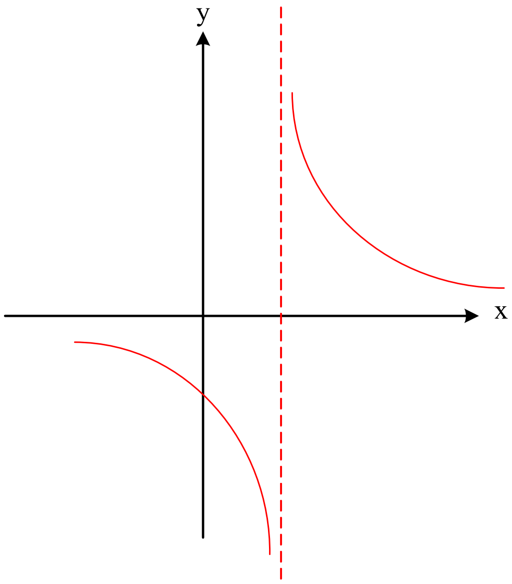
$$\because \lim_{x\to 0}\frac{\sin 4x}{4x}=1 \quad ; \quad \lim_{x\to 0}\frac{\sin 3x}{3x}=1$$

$$\therefore \text{O.E.}=\lim_{x\to 0}\frac{4}{3}\times 1=\frac{4}{3}$$

(5) Infinite Limit

a. Vertical Asymptotes：

if $\lim_{x \to c} f(x) = \pm\infty$ then $x = c$ is a vertical asymptote



$f(x) = \dfrac{1}{x+1}$

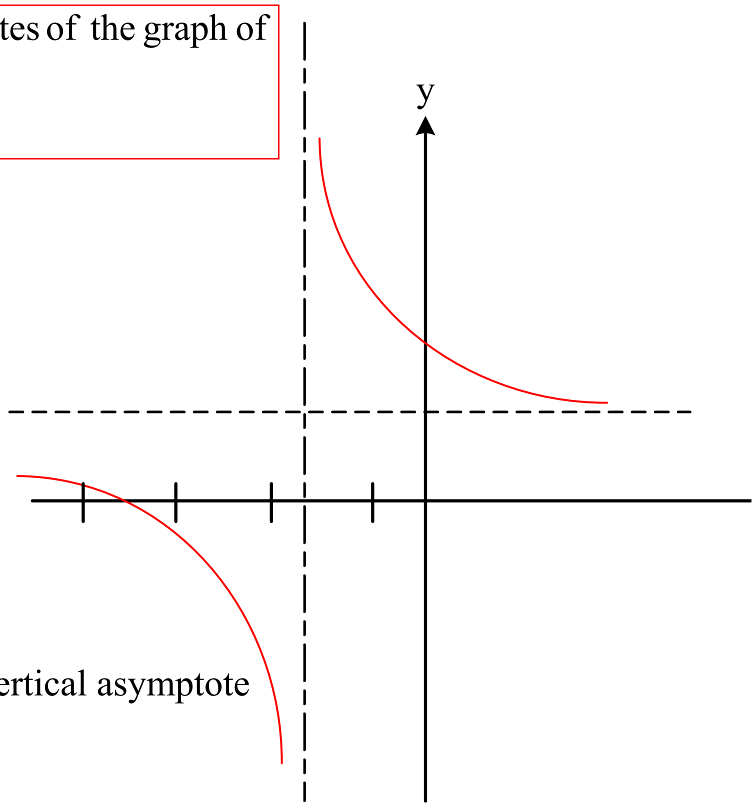
$x = 1$ is a vertical asymptote

> ex.6. Drtermine all vertical asymptotes of the graph of
>
> $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$

Sol：

$f(x) = \dfrac{(x-2)(x+4)}{(x-2)(x+2)} = \dfrac{x+4}{x+2}$,$x \neq 2$

$\lim_{x \to -2^-} f(x) = -\infty$,$\lim_{x \to -2^+} f(x) = \infty$



∴ x=-2 is a vertical asymptote

x= 2 is a discontinue p.t. not a vertical asymptote

b.Horizontal asymptotes

if $\lim_{x \to L} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ then $y = L$ is a Horizontal asymptotes ex.7. Find the horizontal asymptotes of function

$$f(x)=\frac{x^2+2x-8}{x^2-4}$$

Sol： $\lim_{x\to\infty}\frac{x^2+2x-8}{x^2-4}=1$ $\quad\therefore$ $y=1$ is a horizontal asymptotes

ex.8. $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

Sol：

O.E. $=\lim_{x\to\infty}(\infty-\infty)$ is a indeterminate form

$$\lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)}=\lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$

$$=\lim_{x\to\infty}\frac{x/\sqrt{x^2}}{(\sqrt{x^2+x}+x)/\sqrt{x^2}}=\lim_{x\to\infty}\frac{x/x}{\sqrt{(x^2+x)/x^2}+x/x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{2}$$

Chapter 2 Differeniation

- Definition of Differentiation

$$(1)\ f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(x)}{\Delta x}\ldots\ldots\ldots\ldots(1)$$

$$(2)\ f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\ldots\ldots\ldots.(2)$$ Another form

$$\because f'(c)=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}$$ then Let $x=c+\Delta x$ $\Delta x=x-c$ $\Delta x\to 0$ $x\to c$

$$\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

ex.1. Find the derivative w.r.t. x for $y=\frac{2}{x}$
and find the tangent line passes through p.t. (1,2)

Sol：

(1) $$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{2}{x+\Delta x}-\frac{2}{x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{2[x-(x+\Delta x)]}{x(x+\Delta x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{-2\Delta x}{\Delta x(x^2+x\times\Delta x)}$$

$$=\frac{-2}{x^2}$$

(2) $f'(1)=\frac{-2}{1^2}=-2=m$

$(y-y_1)=m(x-x_{1)}$

$y-2=-2(x-1)$

$2x+y-4=0$ $y=-2x+4$

- Derivative of x^n f(x)*g(x)   f(x)/g(x) and Trigonometric Function

1. $\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$
2. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
3. $\frac{d}{dx}\left[f(x)/g(x)\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$
4. $\frac{d}{dx}\left[\sin x\right] = \cos x$ 5. $\frac{d}{dx}\left[\cos x\right] = -\sin x$
6. $\frac{d}{dx}\left[\tan x\right] = \sec^2 x$ 7. $\frac{d}{dx}\left[\cot x\right] = -\csc^2 x$
8. $\frac{d}{dx}\left[\sec x\right] = \tan x \cdot \sec x$ 9. $\frac{d}{dx}\left[\csc x\right] = -\csc x \cdot \cot x$

Sol：

1. $\frac{d}{dx}\left[x^n\right] = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} \cdot x^{n-2} \cdot \Delta x^2 + \ldots\ldots + \Delta x^n) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (n \cdot x^{n-1} + \frac{n \cdot (n-1)}{2} \cdot x^{n-2} \cdot \Delta x + \ldots\ldots\ldots + \Delta x^{n-1}) = n \cdot x^{n-1}$$

2. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x+\Delta x) \cdot g(x) - f(x) \cdot g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot \left[g(x+\Delta x) - g(x)\right]}{\Delta x} + \lim_{\Delta x \to 0} \frac{\left[f(x+\Delta x) - f(x)\right] \cdot g(x)}{\Delta x}$$

$$= f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

4. $\frac{d}{dx}\left[\sin x\right] = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin x \cdot \operatorname{cox} \Delta x + \cos x \cdot \sin \Delta x - \sin x}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1) + \cos x \cdot \sin \Delta x}{\Delta x} = \cos x$$

$$6.\ {}^{d}\!/\!_{dx}\left[\tan x\right] = {}^{d}\!/\!_{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$8.\ {}^{d}\!/\!_{dx}\left[\sec x\right] = {}^{d}\!/\!_{dx}\left[\frac{1}{\cos x}\right] = \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} = \tan x \cdot \sec x$$

ex.2. $f(x) = (3x - 2x^2) \cdot (5 + 4x)$,find $f'(x)$

Sol: $f'(x) = (3x - 2x^2)4 + (5 + 4x)(3 - 4x)$

$$= 12x - 8x^2 + 15 - 20x + 12x - 16x^2 = 15 + 4x - 24x^2$$

ex.3. ${}^{d}\!/\!_{dx}\left[\frac{5x - 2}{x^2 + 1}\right]$

Sol: ${}^{d}\!/\!_{dx}\left[\frac{5x - 2}{x^2 + 1}\right] = \frac{(x^2 + 1) \cdot 5 - (5x - 2) \cdot 2x}{(x^2 + 1)^2} = \frac{5x^2 + 5 - 10x^2 + 4x}{(x^2 + 1)^2}$

$$= \frac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$$

- The Chain Rule

$${}^{d}\!/\!_{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

ex.4. $f(x) = x^2 \cdot \sqrt{1 - x^2}$

Sol: $f'(x) = x^2 \cdot \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(1 - x^2) + \sqrt{1 + x^2} \cdot 2x$

$$= \frac{x^2}{2\sqrt{1 - x^2}} \cdot (-2x) + 2x \cdot \sqrt{1 - x^2}$$

$$= \frac{-x^3}{\sqrt{1 - x^2}} + 2x \cdot \sqrt{1 - x^2}$$

ex.5. $y = \cos^2 3x$,$y' = ?$

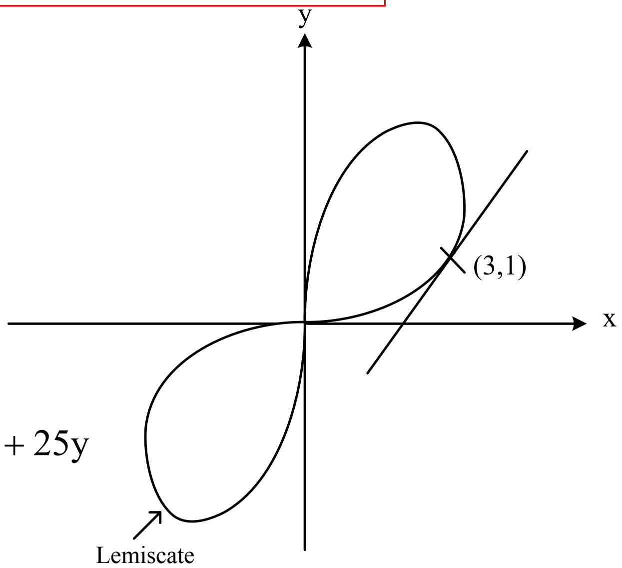
Sol : $y = (\cos 3x)^2$ $\quad \therefore y' = 2 \cdot \cos 3x \cdot (-\sin 3x) \cdot 3 = -6 \cdot \sin 3x \cdot \cos 3x$

- Implicit Differentiation

  Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation w.r.t. x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx

ex.1. Determine the slope of the graph of $3(x^2+y^2)^2=100xy$ at the point $(3,1)$, find the tangent line.



Sol：

$$\frac{d}{dx}\left[3(x^2+y^2)^2\right]=\frac{d}{dx}\left[100xy\right]$$

$$6(x^2+y^2)\cdot(2x+2y\cdot\frac{dy}{dx})=100(x\cdot\frac{dy}{dx}+y)$$

$$3(x^2+y^2)\cdot(x+y\cdot\frac{dy}{dx})=25(x\cdot\frac{dy}{dx}+y)$$

$$3(x^2+y^2)\cdot x+3y\cdot(x^2+y^2)\cdot\frac{dy}{dx}=25x\cdot\frac{dy}{dx}+25y$$

$$\frac{dy}{dx}\left[3y(x^2+y^2)-25x\right]=25y-3x\cdot(x^2+y^2)$$

$$\therefore\frac{dy}{dx}=\frac{25y-3x(x^2+y^2)}{3y(x^2+y^2)-25x}$$

$$m=\frac{29-9(9+1)}{3\cdot(9+1)-25\times 3}=\frac{-65}{-45}=\frac{13}{9}$$

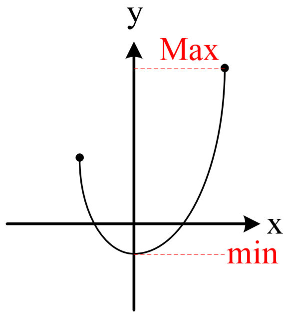
$$(y-1)=\frac{13}{9}(x-3)$$

$$9y-9=13x-39$$

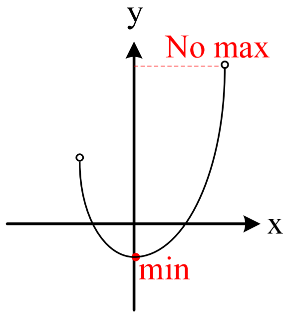
$$13x-9y-30=0$$

Chapter 3 Applications of Differeniation

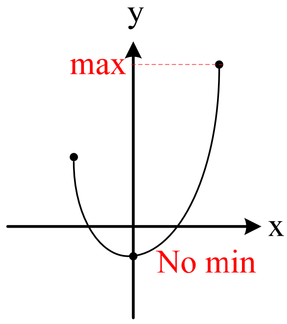
- Extrema on an interval



Close interval



open interval



closed interval

Note：If f(x) is continuous on a closed interval [a,b] ,then f has both a minimum and maximum on the interval.

- Critical number and relative Exterma

1. $f'(c)=0$ or $f'(c)$ is undefined at $x=c$, then c is a critical number.

2. Relative extrema occurs at critical number.

ex.1. Find the extrema of $f(x)=2x-3x^{2/3}$ on the $[-1,3]$

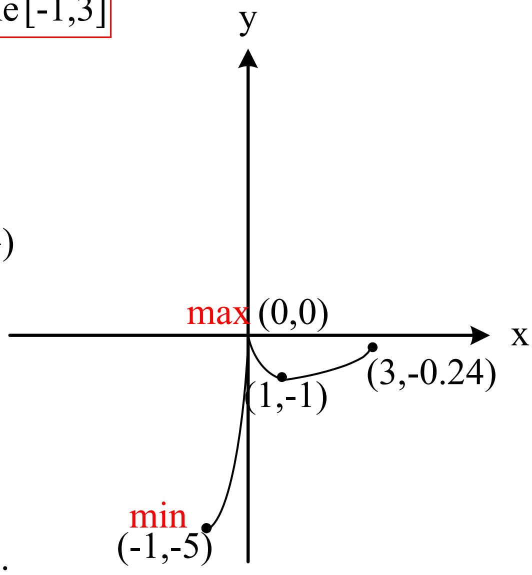
Sol：

(1) $f'(x)=2-3\times\frac{2}{3}\cdot x^{-\frac{1}{3}}=2\cdot(1-\frac{1}{x^{\frac{1}{3}}})=2\cdot(\frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{3}}})$

(2) the critical number :

$f'(x)=0 \quad \therefore x^{\frac{1}{3}}-1=0 \quad \therefore x=1$

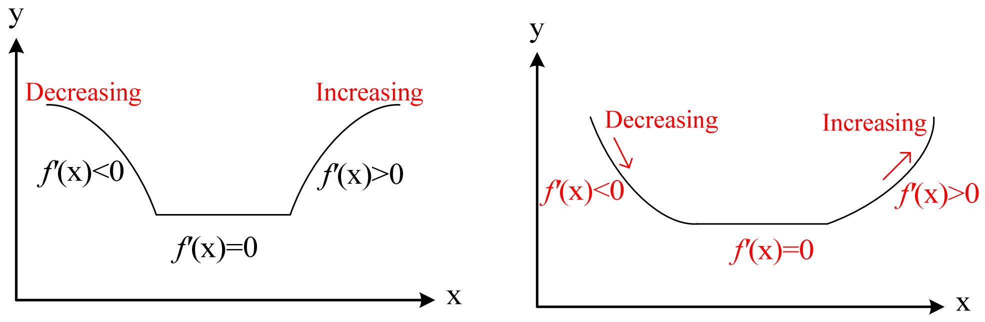
$f'(x)=\text{undefined} \quad \therefore x^{\frac{1}{3}}=0 \quad \therefore x=0$



(3) Evaluate f at critical number and endpoints.

Left endpoint	critical number	critical number	Right endpoint
$f(-1)=-5$	$f(0)=0$	$f(1)=-1$	$f(3)=6-\sqrt[3]{9}=0.24$
Absolute Min	Absolute Max	Relative Min	

- Increasing and Decreasing Functions and First Derivative Test



1. Increasing and Decreasing
   (1) If $f'(x) > 0$ for all x in (a, b), then $f(x)$ is increasing on $[a, b]$
   (2) If $f'(x) < 0$ for all x in (a, b), then $f(x)$ is decreasing on $[a, b]$
2. Guidelines for Find Intervals on which a Function is Increasing or Decreasing.
   (1) Locate the critical numbers of f in (a, b), and use these unmber to determine test intervals.
   (2) Determine the sign of $f'(x)$ at one test value in each of the interval.
   (3) Determine whether f is increasing or decreasing on each interval.

ex.2. Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Sol：

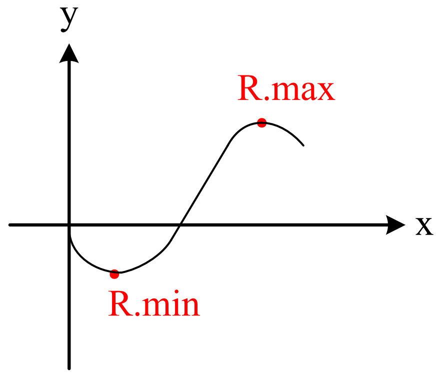
$f'(x) = 3x^2 - 3x = 0 \quad 3x(x - 1) = 0 \qquad \therefore$ Critical number are $x = 0$ and $x = 1$

- First derivative test

let C is a Critical number of a function f, then f(c) can be Classifeid as follows:
(1)If f'(x) Changes from negative to positive at c, then f(c) is a relative minimum.
(2)If f'(x) Changes from positive to negative at c: then f(c) is a relative maximum.

ex.3. Applying the first derivative test to find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$

Sol：



(1) Find the critical numbers

$f'(x) = \frac{1}{2} - \cos x = 0 \qquad \therefore \cos x = \frac{1}{2}$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$

(2)

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	(-)	(+)	(-)
Conclusion	↘	↗	↘

● Concavity

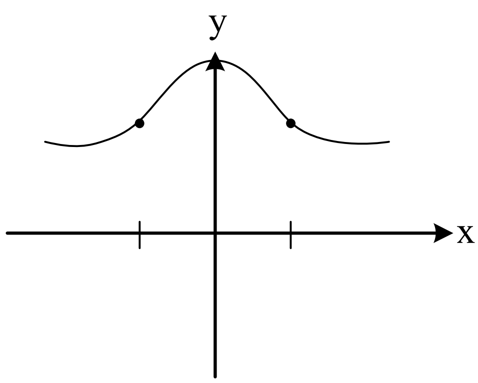
$f'(x)$ is increasing meaning $f''(x) > 0$
$f'(x)$ is decreasing meaning $f''(x) < 0$, therefore
1. $f''(x) > 0$ the graph is concave upward in interval
2. $f''(x) < 0$ the graph is concave downward in interval
3. $f''(x) = 0$ the graph is linear in interval

- Guideline for finding concavity of a function.

1. Locate x value at which f'’(x)=0 or f’’(x) does not exist.
2. Use these value to determine test intervals.
3. Test the sign of f’’(x) in each of the test intervals.

ex.4. Determine the open intervals on which the graph of f(x)
$f(x) = \dfrac{6}{x^2 + 3}$ is concave upward or downward.

Sol：



(1) $f(x) = 6 \cdot (x^3 + 3)^{-1}$ $\therefore f'(x) = -6(x^2 + 3)^{-2} \cdot 2x = -12x(x^2 + 3)^{-2}$

$$f''(x) = -12\frac{(x^2 + 3)^2 \cdot 1 - x \cdot 2(x^2 + 3) \cdot 2x}{(x^2 + 3)^4}$$

$$x = -12\frac{(x^2 + 3)(x^2 + 3 - 4x^2)}{(x^2 + 3)^4} = 36\frac{x^2 - 1}{(x^2 + 3)^3}$$

$\therefore f''(x) = 0$ or does not exist $x = \pm 1$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	(+)	(-)	(+)
Conclusion	$\cup$	$\cap$	$\cup$

- Points of Inflecton

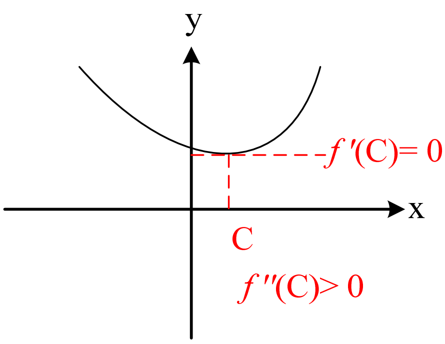
(1) The concavity of f changes at a point of inflection.

(2) If (c,f(c)) is a point of inflection of graph of f , then either f'’(c) = 0 or f'’(c) is undefined.

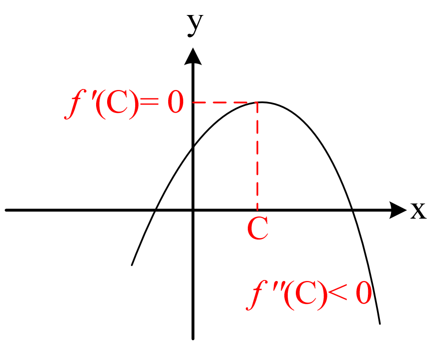
- Second derivative Test

If $f'(c)=0$, then
(1) $f''(c)>0$, then $f(c)$ is minimum.
(2) $f''(c)<0$, then $f(c)$ is maximum.
(3) $f''(c)=0$, then fails. $f(c)$ may be the max. min. or neither.

Note： In case (3) , you can use the First Derivative test



f(x) is a min



f(x) is a max

ex.5. Find the velative extrema for $f(x)=-3x^5+5x^3$

Sol：

(1) $f'(x) = -15x^4 + 15x^2 = 0$

$-15x^2(x^2 - 1) = 0 \quad \therefore x = 0, -1, 1$ are critical numbers

$f''(x) = -60x^3 + 30x^2$

test: (1) $f''(-1) = (+)$ $\quad f(-1) = -2$ is a min

(2) $f''(0) = 0$ $\quad f(0) = 0$ tests fail

(3) $f''(1) = (-)$ $\quad f(1) = 2$ is a max

use First derivative extrema for $x = 0$

Interval	$-1 < x < 0$	$0 < x < 1$
Test value	$x = -0.5$	$x = 0.5$
Sign of $f'(x)$	(-) increased	(+)increasing
Sign of $f''(x)$	$\cap$	$\cup$

$\therefore x = 0$ is a inflection point

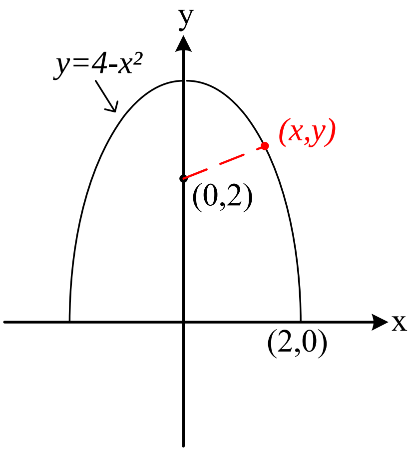
- Optimization problem.

  Guidelines for solving Applied Min. and Max. problems.

1. Identify all given guantities. If possible , make a sketch.
2. Write a primary equation.
3. Reduce the primary equation to one having a single independent variable.
4. Determine the feasible domain of the primary equation.
5. Determine the desired max. and min.

ex.6. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,2)$

Sol：



$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

$$d^2 = x^2 + (4 - x^2 - 2)^2$$

$$= x^2 + (x - 2)^2$$

$$f' = 0 \quad \therefore 2x + 2(2 - x^2)\cdot(-2x) = 0$$

$$x - 4x + 2x^3 = 0$$

$$x(2x^2 - 3) = 0 \qquad x = 0\ , x = \pm\sqrt{\frac{3}{2}}$$

x	$-\sqrt{3/2}$	0	$-\sqrt{3/2}$
	1.329 min	0	1.329 min

- Differentials , Error propagation , and Approximation

$$(1)\ f(x+\Delta x) - f(x) = \Delta y \quad \Rightarrow \quad dy = f'(x)\cdot dx$$
$$\therefore f(x+\Delta x) - f(x) = f'(x)\cdot dx$$
$$(2)\ f(x+\Delta x) = f(x) + f'(x)\cdot dx$$

ex.7. A ball is measured to be 0.7″ , if measurement is correct to within 0.01″ , estimate the propagated error in the volume V.

Sol：

$$V = f(r) = \frac{4}{3}\pi r^2$$
$$dV = f'(r)\cdot dr = 4\cdot \pi r^2 \cdot dr = 4\pi \times 0.7 \times (\pm 0.01'') = \pm 0.06158\ in^3$$
$$\therefore \text{ the volum has a propagated error of } 0.06158\ in^3$$
$$dv/d = \pm 0.06158 \Big/ (\frac{4}{3}\pi \times 0.7^3) = \pm 0.0429 = 4.29\ \%$$

ex.8. Use differentials to approximate $\sqrt{16.5}$

Sol：

$$\text{let } f(x) = \sqrt{x} \qquad x = 16 \qquad dx = 0.5$$

$$f(x + \Delta x) = f(x) + f'(x) \cdot dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot dx$$

$$= \sqrt{16} + \frac{1}{2\sqrt{16}} \cdot 0.5 = 4 + \frac{1}{16} = 4.0625$$

The true value $\sqrt{16.5} = 4.0620$

Chapter 4 Integration

$$dy/dx = f(x)$$

$$dy = f(x) \cdot dx$$

$$y = \int f(x) \cdot dx = F(x) + c$$

- Basic integration rule.
- Indefinite and definite integration

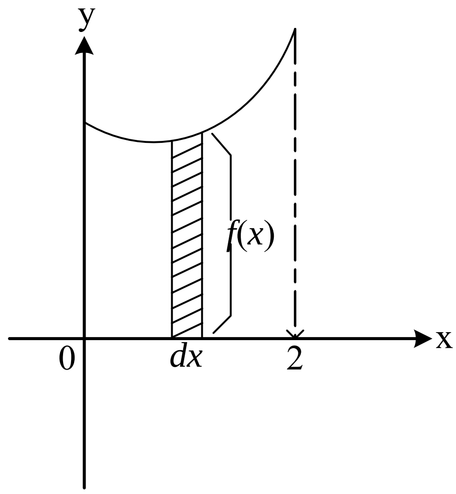
ex.1. a. $\int \frac{1}{x^3} \cdot dx = \int x^{-3} \cdot dx = \frac{x^{-2}}{-2} + c = \frac{1}{-2x^2} + c$

b. $\int \sqrt{x} \cdot dx = \int x^{\frac{1}{2}} \cdot dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} x^{\frac{3}{2}} + c$

c. $\int 2 \cdot \sin x \cdot dx = 2 \cdot \int \sin x \cdot dx = 2 \cdot (-\cos x) + c$

ex.2. find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$

Sol：



$$A = \int_0^2 f(x)\cdot dx = \int_0^2 (2x^2 - 3x + 2)\cdot dx$$

$$= \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^2$$

$$= \left(\frac{2}{3}\times 8 - \frac{3}{2}\times 4 + 4\right) - (0 - 0 + 0)$$

$$= \frac{10}{3}$$

- Integration by Substitution

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))\cdot g'(x)$$

$$d\left[F(g(x))\right] = F'(g(x))\cdot g'(x)\cdot dx$$

$$F(g(x)) = \int F'(g(x))\cdot g'(x)\cdot dx = \int f(g(x))\cdot g'(x)\cdot dx$$

if $u = g(x)$ $\quad du = g'(x)\cdot dx \quad \therefore F(u) = \int f(u)\cdot du$

ex.3. $\int x\cdot(x^2+1)^2\cdot dx$

Sol：

let $u = (x^2+1) \quad du = 2x\cdot dx \Rightarrow x\cdot dx = \frac{du}{2}$

$$O.E. = \int u^2\cdot\frac{du}{2} = \frac{1}{2}\cdot\frac{u^3}{3} + c = \frac{(x^2+1)^3}{6} + c$$

ex.4. $\int x\cdot\sqrt{2x-1}\cdot dx$

Sol：

let $u = 2x - 1$ $du = 2x \cdot dx$ $dx = \frac{du}{2}$ $x = \frac{u+1}{2}$

$$\therefore O.E. = \int \frac{u+1}{2}(u)^{\frac{1}{2}} \cdot \frac{du}{2} = \frac{1}{4} \cdot \int (u^{\frac{3}{2}} + u^{\frac{1}{2}}) \cdot du$$

$$= \frac{1}{4}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) = \frac{1}{10}u^{\frac{5}{2}} + \frac{1}{6}u^{\frac{3}{2}} + c$$

$$= \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c$$

ex.5. $\int_0^1 x \cdot (x^2+1)^3 \cdot dx$

Sol：

let $u = x^2 + 1$ $x = 0, u = 1$ & $x = 1, u = 2$

$du = 2x \cdot dx$ $\therefore x \cdot dx = \frac{du}{2}$

$$O.E. = \int_1^2 u^3 \cdot \frac{du}{2} = \frac{1}{8}u^4 \Big|_1^2 = \frac{1}{8}(16-1) = \frac{15}{8}$$